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The $\hat{c} = \frac{2}{3}$ minimal $N = 1$ superconformal system and its realisation in the critical O(2) Gaussian model†

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Abstract. The structure of the $\hat{c} = \frac{2}{3}$ minimal $N = 1$ superconformal system is analysed in detail. The primary operators are constructed as operators in the critical O(2) Gaussian model at some specific fixed radius. The operator algebra is verified explicitly. Operator product coefficients and some superspace correlation functions are calculated exactly.

1. Introduction

There has been much progress recently concerning two-dimensional critical phenomena and the whole structure of conformal invariance in two dimensions [1]. Supersymmetry also seems to play some role in certain critical models [2-4]. Techniques have been advanced to deal with the calculation of correlation functions and the representation content of critical models.

The purpose of this paper is to analyse extensively a particular superconformal minimal theory and to show that it is fully realised in a certain critical system, namely the O(2) Gaussian model at a point on the critical Kosterlitz-Thouless line.

As shown by Friedan *et al* [5], when the central charge of the $N = 1$ superconformal algebra is less than one, there exists a discrete infinity of unitary superconformal theories. They contain a finite set of operators, which closes into itself under the operator product expansion (OPE). The first non-trivial model in this series has been identified with the tricritical Ising model [2].

The unitary superconformal models with $\hat{c} < 1$ are given by [5]:

$$\hat{c} = 1 - \frac{8}{m(m+2)} \qquad m = 2, 3, 4, \ldots . \tag{1.1}$$

There are two sectors, the NS sector where the fermionic components of superfields are single valued on the complex plane and the R sector where the fermionic components are double valued around the spin fields, which generate the R ground states from the NS vacuum [2]. The finite set of primary operators contained in these theories has dimensions given by

$$\Delta_{p,q} = \frac{[(m+2)p - mq]^2 - 4}{8m(m+2)} \qquad 1 \leq p < m, 1 \leq q < m+2, (p-q) \text{ even} \tag{1.2a}$$

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in the NS sector and

$$\Delta_{p,q}=\frac{[(m+2)p-mq]^2-4}{8m(m+2)}+\frac{1}{16} \qquad 1\leqslant p<m, 1\leqslant q<m+2, (p-q) \text{ odd} \tag{1.2b}$$

in the R sector.

The system that we will deal with has a central charge given by the $m=4$ term in the series [5]. It has a Virasoro anomaly $c=\frac{3}{2}\hat{c}=1$ which coincides with a c of the critical Gaussian model. The dimensions of its primary operators are shown in table 1. In fact, the set of the primary operators in the $\hat{c}=\frac{2}{3}$ theory, including the twisted sector, coincides with the content of the first minimal model of the $N=2$ superconformal algebra. This was proven rigorously in [6] using the characters of the $N=2$ superconformal algebra.

Table 1. Dimensions of the primary operators of our model, in NS and R sectors.

NS	R
0	$\frac{1}{24}$
1	$\frac{1}{16}$
$\frac{1}{6}$	$\frac{9}{16}$
$\frac{1}{16}$	$\frac{3}{8}$

In [3] it was shown that the $N=2$ superconformal algebra is realised in the critical O(2) Gaussian model at a fixed radius.

Below we show that in fact the full $\hat{c}=\frac{2}{3}$ superconformal model is realised in the critical Gaussian model. We will be discussing the $N=1$ formalism since it is easier to deal with. Later on we will discuss how we can assemble the $N=2$ structure from the $N=1$ one. The O(2) Gaussian model is described at the critical line by a single free-scalar field. The remaining piece that needs to be added is the 'twist' field of the scalar field. This is known to describe the twisted sector of the first $N=2$ superconformal minimal model [6]. The strategy is to construct explicitly all the primary operators in the $\hat{c}=\frac{2}{3}$ model from vertex operators and the twist family, and then show that they obey the correct operator algebra implied by the structure of the $N=1$ superconformal algebra [8]. Some correlation functions will be also calculated and shown to be supermeromorphic functions in $N=1$ superspace.

The structure of this paper is as follows. In § 2 we explicitly construct the 'untwisted' primary operators of the NS and R sectors and we verify the respective operator algebra. In § 3 the 'twisted' set of operators are constructed and the relevant fusion rules verified. Some correlation functions are also calculated as functions in $N=1$ superspace. We also discuss the possible modular invariant models by 'sewing' together left and right sectors. Finally § 4 contains remarks concerning the realisation of the $\hat{c}=\frac{2}{3}$, $N=1$ superconformal system in the critical XY and AT models as well as our conclusions.

2. Construction of the untwisted operators

At the critical point the O(2) Gaussian model is described by a two-dimensional vector. We will consider the radius of this vector to be fixed to $R=\sqrt{3}$. The only remaining degree of freedom is the phase which is a free-scalar field $\phi(\sigma)$ and for fixed time τ it provides a map $\phi(\sigma): S^1 \to S^1$. This map is periodic in σ, i.e. $\phi(\sigma+2\pi)=\phi(\sigma)$, and the target space is a circle due to the fact that $\phi(\sigma)$ is a phase. The operators in this theory are the 'vertex' operators, $V_a(\sigma) \equiv : \exp(\mathrm{i}a\phi(\sigma))$: and their derivatives. In order for these operators to be well defined in the target space, a must take the values: $a=n/R$, $n \in \boldsymbol{Z}$ or $\boldsymbol{Z}\oplus\frac{1}{2}$, where R is the radius†. Integer values for n correspond to bosonic operators, while half-integer ones correspond to fermionic operators. We will work from now on with holomorphic coordinates z and $\bar{z}$. At the critical point the z and $\bar{z}$ dependence factorises. We will discuss the left sector of the theory which depends on z only. The discussion of the right sectors parallels that of the left one.

The two-point function of $\phi(z)$ is given by:

$$\langle 0|\phi(z)\phi(w)|0\rangle = -\ln(z-w) \tag{2.1}$$

where radial quantisation is assumed and the vacuum is the SL(2, C) invariant vacuum.

The energy-momentum tensor is of the standard form, $T(z) \equiv -\frac{1}{2}: \partial_z\phi(z)\partial_z\phi(z)$: with an OPE‡

$$T(z)T(w)=\frac{1}{2(z-w)^4}+\frac{2T(w)}{(z-w)^2}+\frac{\partial_w T(w)}{(z-w)}. \tag{2.2}$$

A vertex operator $V_a(z)$ has dimension $\Delta=a^2/2$ as shown by the following OPE:

$$T(z)V_a(w)=\frac{a^2}{2}\frac{V_a(w)}{(z-w)^2}+\frac{\partial_w V_a(w)}{(z-w)}. \tag{2.3}$$

To find the superpartner of $T(z)$, we have to find an operator with $\Delta=\frac{3}{2}$. There are two candidates, $V_{\sqrt{3}}(z)$ and $V_{-\sqrt{3}}(z)$, as well as any linear combination of the two, which has the correct dimension. For reasons that will be explained below, the correct form is

$$G(z) \equiv (\mathrm{i}/\sqrt{3})[: \exp(\mathrm{i}\sqrt{3}\phi(z)) :-: \exp(-\mathrm{i}\sqrt{3}\phi(z)) :]. \tag{2.4}$$

Then the $N=1$ superconformal algebra closes correctly

$$G(z)G(w)=\frac{2}{3}\frac{1}{(z-w)^3}+\frac{2T(w)}{(z-w)}. \tag{2.5}$$

The primary operators in the NS sector are generated by primary superfields acting on the NS vacuum. A superfield is a function in superspace§: $\Phi(z)=g(z)+\theta\psi(z)$. If Δ is the dimension of the bosonic components $g(z)$, then the corresponding dimension for the fermionic partner $\psi(z)$ is $\Delta+\frac{1}{2}$. A primary superfield operator is defined through the following OPE with the super-energy-momentum tensor

$$\begin{aligned} G(z)g(w)&=\frac{\psi(w)}{(z-w)} & T(z)g(w)&=\frac{\Delta g(w)}{(z-w)^2}+\frac{\partial_w g(w)}{(z-w)} \\ G(z)\psi(w)&=\frac{\partial_w g(w)}{(z-w)}+2\frac{g(w)}{(z-w)^2} & T(z)\psi(w)&=(\Delta+\tfrac{1}{2})\frac{\psi(w)}{(z-w)^2}+\frac{\partial_w\psi(w)}{(z-w)}. \end{aligned} \tag{2.6}$$

† In our case, $R=\sqrt{3}$.

‡ We will always suppress non-singular contributions to the OPE.

§ z and θ are the coordinates in superspace denoted collectively with

The obvious candidate for the $\Delta = 1$ primary operator is the U(1) current of the system, $g_1(z) \equiv \mathrm{i}\partial_z\phi(z)$. It has an OPE with $G(z)$:

$$G(z)g_1(w) = -\mathrm{i}\left(\frac{:\exp(\mathrm{i}\sqrt{3}\phi(w)):+:\exp(-\mathrm{i}\sqrt{3}\phi(w)):}{(z-w)}\right). \tag{2.7}$$

From (2.6) we can infer that the superpartner of $g_1(z)$ is $\psi_1(z) = -\mathrm{i}[V_{\sqrt{3}}(z) + V_{-\sqrt{3}}(z)]$. It is an easy exercise to check that the rest of the relations (2.6) are satisfied.

$G(z)$ and $\psi_1(z)$ are the two supercharges of the corresponding $N = 2$ minimal theory which, along with $T(z)$ and $g_1(z)$, complete the $N = 2$ super-energy-momentum tensor multiplet [3, 4]. As far as the other presentations are concerned they can be built from the $N = 1$ representations without adding new fields in the supermultiplet. As was shown in [4], for $N = 2$ representations degenerate at level $\frac{1}{2}$ one of the fermionic components vanishes identically while the second bosonic component is the derivative of the first one. Thus the $N = 2$ supermultiplets contain the same number of degrees of freedom as the $N = 1$ supermultiplets. Using the remarks above the $N = 2$ structure can be easily reconstructed from the $N = 1$ structure.

There are two $\Delta = \frac{1}{6}$ operators with opposite U(1) charge: $g^{\pm}_{1/6}(z) \equiv \exp[\pm(\mathrm{i}/\sqrt{3})\phi(z)]$. Using the following relation:

$$:\exp(\mathrm{i}a\phi(z))::\exp(\mathrm{i}b\phi(w)):=(z-w)^{ab}:\exp(\mathrm{i}a\phi(z)+\mathrm{i}b\phi(w)): \tag{2.8}$$

we can calculate the superpartners of $g^{\pm}_{1/6}(z)$: $\psi^{\pm}_{1/6} = \mp\mathrm{i}/\sqrt{3}:\exp[\mp(2\mathrm{i}/\sqrt{3})\phi(z)]$. As can be seen from figure 1, apart from the $\Delta = \frac{1}{16}$ operator, the above exhaust the set of primary operators in the NS sector. Correlation functions of vertex operators can be calculated through the standard formula

$$\langle 0|\prod_{i=1}^{n} V_{a_i}(z)|0\rangle = \prod_{i<j}^{n}(z_{ij})^{a_ia_j}\delta\left(\sum_i a_i\right). \tag{2.9}$$

The sum of the coefficients a_i in (2.9) has to be zero, otherwise IR divergences force the correlation function to vanish [7]. To calculate the operator algebra of the operators above, one has to calculate the appropriate three-point functions. The idea is that if $[\Phi_1]\otimes[\Phi_2] \sim [\Phi_3]$, then $\langle\Phi_1\Phi_2\Phi_3\rangle \neq 0$. The only non-trivial three-point functions that are non-zero are given below:

$$\langle g_1(z_1)g_1(z_2)T(z_3)\rangle = (z_{13}z_{23})^2 \qquad \langle g^{+}_{1/6}(z_1)g^{-}_{1/6}(z_2)T(z_3)\rangle = \tfrac{1}{6}z_{12}^{5/3}(z_{13}z_{23})^{-2} \tag{2.10a}$$

$$\langle g^{+}_{1/6}(z_1)g^{-}_{1/6}(z_2)g_1(z_3)\rangle = \frac{1}{\sqrt{3}}z_{12}^{2/3}(z_{13}z_{23})^{-1}$$

$$\langle g^{+}_{1/6}(z_1)g^{+}_{1/6}(z_2)\psi^{+}_{1/6}(z_3)\rangle = -\frac{\mathrm{i}}{\sqrt{3}}\left(\frac{z_{12}}{z_{13}^2z_{23}^2}\right)^{1/3}. \tag{2.10b}$$

Relations (2.10) imply the following operator algebra:

$$[1]\otimes[1] = [0] \qquad [1]\otimes[\tfrac{1}{6}] = [\tfrac{1}{6}] \tag{2.11a}$$

$$[\tfrac{1}{6}]\otimes[\tfrac{1}{6}] = [0] + \alpha_1[1] + \alpha_2[\tfrac{1}{6}+\tfrac{1}{2}] \qquad \alpha_1 = \frac{1}{\sqrt{3}} \qquad \alpha_2 = -\frac{\mathrm{i}}{\sqrt{3}} \tag{2.11b}$$

which is in accord with the known 'fusion rules' [8].

In the Ramond sector the two ground states are generated from the NS vacuum by the corresponding spin-field operators, $\Theta(z)$ and $\bar{\Theta}(z)$ of dimension $\Delta = \frac{1}{24}$, $G_0|\Theta\rangle = |\bar{\Theta}\rangle$.

One of them, $|\bar{\Theta}\rangle$ is degenerate at level zero and thus decouples. Correspondingly, $G(z)\Theta(w)\sim O[(z-w)^{1/2}]$. $\Theta(z)$ can be also represented as a vertex operator:

$$\Theta^{\pm(z)}:=\exp\left(\pm\frac{\mathrm{i}}{2\sqrt{3}}\phi(z)\right).$$

We can explicitly compute

$$G(z)\Theta^{\pm}(w)=\mp\frac{\mathrm{i}}{\sqrt{3}}:\exp\left(\mp\mathrm{i}\frac{5}{2\sqrt{3}}\phi(w)\right):\frac{1}{(z-w)^{1/2}}. \tag{2.12}$$

As expected, $\Theta^{\pm}(z)$ create cuts in the complex plane around which the fermionic components of the superfields are double valued. The Ramond primary operators are generated from the Ramond ground state by the action of superfield operators.

The operator of dimension $\Delta=\frac{3}{8}$ in the R sector can be represented by $g^{\pm}_{3/8}(z)=$: $\exp(\pm\mathrm{i}\frac{1}{2}\sqrt{3}\phi(z))$:. It is generated by the superfield operator $\Phi_{1/6}(z)$ acting on the R vacuum. We can explicitly verify the following OPE:

$$\begin{aligned}
&[0+\tfrac{3}{2}]\otimes[\tfrac{1}{24}]_+=[\tfrac{1}{24}+1]_+ && [0+\tfrac{3}{2}]\otimes[\tfrac{3}{8}]_-=[\tfrac{3}{8}]_+\\
&[\tfrac{1}{6}]_+\otimes[\tfrac{1}{24}]_+=[\tfrac{3}{8}]_+ && [\tfrac{1}{6}]_+\otimes[\tfrac{1}{24}]_-=[\tfrac{1}{24}]_-\\
&[\tfrac{1}{6}+\tfrac{1}{2}]_+\otimes[\tfrac{1}{24}]_+=[\tfrac{3}{8}]_- && [\tfrac{1}{6}+\tfrac{1}{2}]_+\otimes[\tfrac{1}{24}]_-=[\tfrac{1}{24}+1]_+.
\end{aligned} \tag{2.13}$$

By replacing $+\leftrightarrow-$, (2.13) remains valid.

The operators constructed so far correspond to all the operators of the NS and R sectors of the corresponding $N=2$ model.

3. Construction of the twisted operators

In this section we introduce the notion of a 'twist' field and use it to construct the remaining operators in the model.

In [6] it was shown that the single operator of the twisted sector of the $\tilde{c}=\frac{1}{3}$, $N=2$ model with $\Delta=\frac{1}{16}$ decomposes into the $\Delta=\frac{1}{16}$ operator in the NS sector of the $N=1$ system. Since the operator in the T sector twists one of the two bosonic components of the $N=2$ superfields, it is natural to expect that a candidate for the $\Delta=\frac{1}{16}$ operator is the 'twist' field $H^{\pm}(z)$, which twists the scalar field $\phi(z)$†.

A twisted scalar field can be defined as a map: $S^1\to S^1$ which is antiperiodic in σ:

$$\phi(\sigma+2\pi)=-\phi(\sigma). \tag{3.1}$$

There are two twist fields, $H^{\pm}(z)$, corresponding to the two fixed points of the map (3.1), one at zero and the other at πR. Correlation functions of twist fields are invariant under any of the following three transformations: $H^+\to-H^+$, $H^-\to-H^-$, $H^+\leftrightarrow H^-$. A twist field at $z=0$ and another one at $z=\infty$ generate a cut in the complex plane. $\phi(z)$ transported around a closed contour encircling $z=0$, picks up a minus sign. In the presence of twist fields $\phi(z)$ has a different two-point function [9]:

$${}_{\mathrm{T}}\langle 0|\phi(z)\phi(w)|0\rangle_{\mathrm{T}}\equiv\frac{\langle 0|H^-(\infty)\phi(z)\phi(w)H^+(0)|0\rangle}{\langle 0|H^-(\infty)H^+(0)|0\rangle}=\ln\left(\frac{\sqrt{z}+\sqrt{w}}{\sqrt{z}-\sqrt{w}}\right). \tag{3.2}$$

† A similar idea has been also advocated by D Friedan.

The operator $\partial_z\phi(z)$ is double valued in the presence of spin fields, information which is encoded in the OPE:

$$\partial_z\phi(z)H^{\pm}(w)=\tau^{\pm}(w)/(z-w)^{1/2} \tag{3.3}$$

where $\tau^{\pm}(w)$ are excited twist fields with dimension differing by $\frac{1}{2}$ from that of $H^{\pm}(z)$. To find explicitly the dimension of $H^{\pm}(z)$ we have to calculate $F(z) \equiv {}_{\mathrm{T}}\langle 0|T(z)|0\rangle_{\mathrm{T}}$. Using the explicit form of $T(z)$ and (3.2) we obtain $G(z)=\frac{1}{16}z^{-2}$. Then:

$$\Delta_{\pm}=\frac{1}{2\pi \mathrm{i}}\oint_0 z\,\mathrm{d}z\,F(z)=\tfrac{1}{16}. \tag{3.4}$$

It will be useful to be able to calculate correlation functions of vertex operators in the presence of twist fields. After some straightforward algebra, one obtains

$${}_{\mathrm{T}}\langle 0|\prod_{i=1}^{n} V_{a_i}(z_i)|0\rangle_{\mathrm{T}}=\prod_{i=1}^{n}(2^{-a_i^2}z_i^{-a_i^2/2})\prod_{i<j}^{n}\left(\frac{\sqrt{z_i}+\sqrt{z_j}}{\sqrt{z_i}-\sqrt{z_j}}\right)^{-a_ia_j}. \tag{3.5}$$

Using (3.5) one can ascertain that from the primary operators in the NS sector, only $g_1(z)$ and $G(z)$ have vanishing three-point functions with two twist fields. The three-point function of three twist fields is automatically zero due to 'twist conservation'.

Thus we have the following OPE†:

$$[\tfrac{1}{16}]\otimes[\tfrac{1}{16}]=[0]\oplus \mathrm{i}\frac{\sqrt{3}}{8}[1+\tfrac{1}{2}]\oplus 2^{-1/3}[\tfrac{1}{6}]\oplus[\tfrac{3}{2}]\oplus 2^{1/3}[\tfrac{1}{6}+\tfrac{1}{2}] \tag{3.6}$$

where the coefficients can be found by computing the respective three-point functions. It is now clear why we decided in the beginning to choose a particular linear combination as a candidate for $G(z)$. It had to give the correct OPE (3.6) according to the known 'fusion rules' [8].

The superpartner of $H^{\pm}(z)$ is given by

$$G(z)H^{\pm}(w)=\tilde{H}^{\pm}(w)/(z-w) \qquad \tilde{\Delta}_{\pm}=\tfrac{1}{16}+\tfrac{1}{2}. \tag{3.7}$$

In fact, the form of the null vectors of the $N=2$ T algebra at level $\frac{1}{2}$ [6] imply that the operators $\tau^{\pm}(z)$ and $\tilde{H}^{\pm}(z)$ are identical, something that can be deduced also directly by computing $\langle\tau^{+}\tilde{H}^{-}\rangle$ and finding a non-zero result.

Let us now investigate the operator product $[1]\otimes[\frac{1}{16}]$. Due to twist conservation, the only families that are allowed to appear are $[\frac{1}{16}]$ and $[\frac{1}{16}+\frac{1}{2}]$.

Since the expectation value of $g_1(z)$ in the presence of two twist fields is zero, $[\frac{1}{16}]$ is not present in the operator product. To investigate the appearance of $[\frac{1}{16}+\frac{1}{2}]$ we must find $\langle 0|H^{+}(\infty)\mathrm{i}\partial_z\phi\tilde{H}^{+}(0)|0\rangle$. To evaluate this three-point function, we first compute

$$F(z,w)\equiv\frac{\langle 0|H^{+}(\infty)\mathrm{i}\partial_z\phi(z)G(w)H^{+}(0)|0\rangle}{\langle 0|H^{+}(\infty)H^{+}(0)|0\rangle}=\frac{\mathrm{i}\sqrt{3}}{4}\frac{z^{1/2}}{(z-w)w}. \tag{3.8}$$

Now, if we let $w\to 0$, using (3.8) we can find $\langle 0|H^{+}(\infty)\mathrm{i}\partial_z\phi(z)\tilde{H}^{+}(0)|0\rangle$ as the residue of the $1/\omega$ pole. This gives

$$\frac{\langle 0|H^{+}(\infty)\mathrm{i}\partial_z\phi(z)\tilde{H}^{+}(0)|0\rangle}{\langle 0|H^{+}(\infty)H^{+}(0)|\rangle}=-\frac{\mathrm{i}\sqrt{3}}{4}z^{-3/2}\neq 0. \tag{3.9}$$

† We should remind the reader that the phases of the operator product coefficients are arbitrary. They are irrelevant in a theory with scalar operators.

Consequently $[1]\otimes[\frac{1}{16}]=\mathrm{i}\frac{1}{4}\sqrt{3}[\frac{1}{16}+\frac{1}{2}]$. The only remaining OPE to compute in the NS sector is $[\frac{1}{6}]\otimes[\frac{1}{16}]$. Again, conservation of twist implies that only the families $[\frac{1}{16}]$ and $[\frac{1}{16}+\frac{1}{2}]$ can appear in the operator product. Doing an analogous computation as above we find

$$[\tfrac{1}{6}]\otimes[\tfrac{1}{16}]=2^{-1/3}[\tfrac{1}{16}]\otimes\frac{\mathrm{i}}{2\sqrt{3}}2^{-1/3}[\tfrac{1}{16}+\tfrac{1}{2}] \tag{3.10}$$

in accord with [8]. The picture of the NS sector is now complete.

Moving back again to the R sector, we have to identify the remaining operators of dimension $\frac{1}{16}$ and $\frac{9}{16}$. Let us consider

$$\langle R|H^+(z)H^+(w)|R\rangle\equiv\frac{\langle 0|\Theta^-(\infty)H^+(z)H^+(w)\Theta^+(0)|0\rangle}{\langle 0|\Theta^-(\infty)\Theta^+(0)|0\rangle}. \tag{3.11}$$

We can compute this correlation function by making a Möbius transformation, pushing $z\to 0$, $w\to\infty$ and thus reducing it to a correlation of vertex operators in the twisted vacuum

$$\langle R|H^+(z)H^+(w)|R\rangle=2^{-1/6}[zw(z-w)]^{1/24}\left(\frac{\sqrt{z}+\sqrt{w}}{\sqrt{z}-\sqrt{w}}\right)^{1/12}. \tag{3.12}$$

Letting $w\to 0$ we obtain

$$\lim_{w\to 0}\langle R|H^+(z)H^+(w)|R\rangle=2^{-1/6}w^{-1/24}[1+\tfrac{1}{6}(z/w)^{1/2}+\mathrm{O}(w)] \tag{3.13}$$

which shows that the lowest dimension operator in the OPE $[\frac{1}{16}]_{\mathrm{NS}}\otimes[\frac{1}{24}]$ has $\Delta=\frac{1}{16}$ whereas the next lowest one has $\Delta=\frac{9}{16}$. The operator with dimension $\frac{1}{16}$ is in fact the primary operator generating $[\frac{1}{16}]_{\mathrm{R}}$. We claim that the operator above with dimension $\frac{9}{16}$ is also a primary operator generating $[\frac{9}{16}]_{\mathrm{R}}$. To verify our claim we have to calculate the expectation value of $T(z)$ in the presence of this operator. If this state is primary the singular terms can be at most $\mathrm{O}(z^{-2})$. We find for this correlation function:

$$\begin{aligned}\langle R|H^+(z)T(w)|\tfrac{9}{16}\rangle_{\mathrm{R}}&\sim\frac{z^{1/24}}{w(z-w)}\\&\times\left[\frac{2\sqrt{z}}{(w)}+\frac{1}{6\sqrt{z}}\left(2-\frac{3z}{z-w}+\frac{3z}{w}+\frac{9}{2}\frac{w}{z-w}-\frac{3}{2}\frac{z-w}{w}\right)\right]\end{aligned} \tag{3.14}$$

which shows explicitly that $[\frac{9}{16}]_{\mathrm{R}}$ is primary†.

Thus $[\frac{1}{16}]_{\mathrm{NS}}\otimes[\frac{1}{24}]=[\frac{1}{16}]_{\mathrm{R}}\oplus 2^{-7/12}\sqrt{3}[\frac{9}{16}]_{\mathrm{R}}$ and the construction of the primary operators of the $\hat{c}=\frac{2}{3}N=1$ superconformal system is now complete.

Correlation functions of the operators above can be easily computed. We give here two examples of four-point functions. We use superfields, $\Phi_\Delta(z)=g_\Delta(z)+\theta\psi_\Delta(z)$,

$$\langle 0|\Phi^+_{1/6}(z_1)\Phi^-_{1/6}(z_2)\Phi^+_{1/6}(z_3)\Phi^-_{1/6}(z_4)|0\rangle=(z_{14}z_{23})^{1/3}(u+1)^{1/3}[1+y/3(u+1)] \tag{3.15a}$$

$$\langle 0|\Phi^+_{1/6}(z_1)\Phi^-_{1/6}(z_2)\Phi_1(z_3)\Phi_1(z_4)|0\rangle=z_{12}^{-1/3}z_{34}^{-2}\left(\frac{3u(u+1)+1}{3u(u+1)}-\frac{y}{3}\frac{3u+2}{u(u+1)^2}\right) \tag{3.15b}$$

† This does not guarantee that $[\frac{9}{16}]_{\mathrm{R}}$ is not the superpartner of $[\frac{1}{1}]_{\mathrm{R}}$, but another computation shows that the two operators are in fact orthogonal.

where u, y are OSP $(2|1)$ invariants given by

$$u = \frac{z_{14}z_{23}}{z_{12}z_{34}} \qquad y = u + 1 - \frac{z_{13}z_{24}}{z_{12}z_{34}} \qquad y^2 = 0 \qquad z_{ij} = z_i - z_j - \theta_i\theta_j. \tag{3.16}$$

So far we have been discussing the left sector. The full theory is the tensor product of the left and right sectors. There is no unique way of taking the tensor product though. However, there is a physical principle that restricts the possible ways of sewing together the left and right sectors, and this is modular invariance. If one defines the system on a plane rectangle with periodic boundary conditions (a topological torus), then modular invariance is equivalent to the invariance of the system under global reparametrisations of the torus. Modular invariance puts severe constraints on the operator content of conformal field theories [10]. A similar analysis carries through for the $N = 1$ superconformal models [11]. In our case, there are two modular invariamt combinations. Let us denote the operators by $(\Delta, \bar{\Delta})$, where Δ, $(\bar{\Delta})$ are their dimensions under the left (right) Virasoro algebra. The physical dimension of the operator is $x = \Delta + \bar{\Delta}$ and its spin, $S = \Delta - \bar{\Delta}$. The first solution contains only scalar operators with $\Delta = \bar{\Delta}$, and Δ takes all the possible values of table 1. In fact, this describes two different theories, since there is a sign ambiguity in the partition function of the R operators with antiperiodic boundary conditions in time. The second solution contains the following operators: in the NS sector, $[0, 0]$, $[\frac{1}{6}, \frac{1}{6}]$, and $[1, 1]$, all of spin zero and $[1, 0]$, $[0, 1]$ of spin ± 1; in the R sector, $[\frac{1}{24}, \frac{1}{24}]$ and $[\frac{3}{8}, \frac{3}{8}]$, both being scalar. The second theory is $N = 2$ superconformal invariant and contains both the NS and R sector but not the T sector.

4. Conclusions and remarks

As mentioned previously, the model describes a specific critical point in the O(2) Gaussian model or the XY model. The XY model below the Kosterlitz-Thouless critical temperature T_c flows to the Gaussian critical line with $c = 1$. Where one ends up on this line depends on the specific value of $\beta = J/kT$ that one starts from. Since different radii just rescale β, a fixed radius corresponds to a fixed point on the critical line. There is a whole series of conjectured or proven critical exponents for the $O(n)$ models [7, 12, 13]. From them we can extrapolate to $n = 2$ in which case we recover the exponents at the Kosterlitz-Thouless point. In particular, the thermal exponents $x_{T_n} = n^2/2$ for n even correspond to the $[1, 1]$ family whereas the exponents $x_{H_n} = (2n-1)^2/8$ for $n = 0$, $1 \bmod(4)$ correspond to the $[\frac{1}{16}, \frac{1}{16}]$ family, while for $n = 2, 3 \bmod(4)$ they correspond to the $[\frac{9}{16}, \frac{9}{16}]$ family.

The Gaussian model describes also the AT model along the $\beta = 1$, $-1 \leqslant \lambda \leqslant 1$ critical line†. The AT model is described by two Ising spins coupled with a four-spin interaction. There are two couplings, β governing the strength of the four-spin interaction and λ governing the spin-spin interactions. At $\beta = 1$ the strength of the four-spin interaction vanishes and there is a line of critical points, $-1 \leqslant \lambda \leqslant 1$, of infinite order (what is known as of the Kosterlitz-Thouless type). The critical exponents are varying continuously on the line. The point $\lambda = 0$ corresponds to two decoupled Ising models ($Z_2 \otimes Z_2$ symmetry), whereas at $\lambda = \pm 1$ the model has a Z_4 symmetry corresponding to the critical Potts model (ferromagnetic or antiferromagnetic).

† For more details on the model and its phase diagram we refer the reader to [14]. We will follow the notation of the previous reference.

As mentioned above, the AT model on the critical line is described by a free boson. The action in the continuum limit on the line $\beta = 1$ is

$$S = -K(\lambda) \int d\tau\, d\sigma\, \phi \nabla^2 \phi \tag{4.1}$$

where $K = (2/\pi)(1 - \cos^{-1}(\lambda)/\pi)$. Going to complex coordinates, $\ln(z) = \tau + i\sigma$, it is easy to see that the theory has a factorised z, $\bar{z}$ dependence. The two-point function is

$$\langle 0|\phi(z)\phi(w)|0\rangle = -\frac{1}{4\pi K}\ln(z-w) \tag{4.2}$$

whereas the energy-momentum tensor is given by $T(z) = -(K/2\pi) : \partial_z\phi(z)\partial_z\phi(z) :$, satisfying (2.2). The dimension of the vertex operator $\exp(ia\phi)$ is given by $\Delta_a = a^2/8\pi K$. From that it is obvious that the radius of the boson is given by $R = (6\pi K)^{1/2}$.

The point $\lambda = -\frac{1}{2}\sqrt{2}$ is the Kosterlitz-Thouless point and as we mentioned earlier the system is described by the $\hat{c} = \frac{2}{3}$, $N = 1$ superconformal system (realising also $N = 2$ superconformal invariance [3, 4]). This is supported by the existence in the spectrum of the thermal critical exponent $x_T = 2$, the magnetic exponent $x_H = \frac{1}{8}$, the second magnetic exponent $x_h = \frac{9}{8}$ and a parafermionic operator found in [15], with spin $\frac{1}{2}$ and dimension $\frac{5}{8}$ constant on the whole critical line corresponding to the family $[\frac{9}{16}, \frac{1}{16}]$. If we use the previously mentioned relation between R and K we find $R = \sqrt{3}$ which is what we used in the beginning. By redefining the scalar field and the corresponding vertex operators in the previous sections we can easily map the previous construction to the $\lambda = -\frac{1}{2}\sqrt{2}$ model.

The critical system above is of phenomenological importance since it seems to describe the superfluid-to-normal transition of helium films [16] and possibly critical behaviour in planar magnetics [17] and liquid crystals [18].

To conclude, we constructed the full operator content of the $\hat{c} = \frac{2}{3}$ minimal $N = 1$ superconformal model using the operators of the Gaussian model at a specific point on the critical line. We verified explicitly the corresponding OPE and evaluated some correlation functions. This proves the existence of $N = 1$ superconformal symmetry in the Gaussian model at a fixed radius.

We should stress once more the importance of the conformal approach to critical systems, which will eventually (and hopefully) unify the description of the different universality classes of critical behaviour in two dimensions.

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Node Added. After the completion of this work we received references [19-21] where related issues have been discussed.

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